On the symmetry of the linearized Boltzmann equation

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Abstract. Relations between different problems described by the linearized Boltzmann equation (LBE) is discussed on the basis of the classical symmetry of LBE and kinetic boundary condition. The theory is developed for the system of arbitrary Knudsen number (Kn) from the view point of the Green function. After showing the main idea of the approach to steady problems in bounded and unbounded domain, its extension to unsteady problems in a fixed bounded domain is given. In particular, in the latter, recovery of Kubo formula for the viscosity and thermal conductivity in the fluctuation–dissipation theorem and its extension to the systems of arbitrary Kn is shown. A comment on the extension to gas mixtures is also given.

Keywords: Boltzmann equation, kinetic theory, symmetry, Green reciprocity, Onsager–Casimir relation, fluctuation–dissipation theorem

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INTRODUCTION

Relations between different problems described by the linearized Boltzmann equation have been discussed by many researchers because of the interest in Onsager–Casimir relation based on the entropy production argument (e.g., Refs. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]). For example, Poiseuille flow vs. thermal transpiration and thermophoresis vs. thermal polarization are typical examples of such relations. In the present paper, we review the main results of our recent papers [11, 12, 13] that are based on the idea of the Green function and give a direct interpretation of those relations without entropy production argument. The theory developed in these references makes use of the classical symmetry of the linearized collision operator and scattering kernel of the boundary condition [15, 5], complemented by the discussions on the far field behavior in the case of unbounded domain. The used symmetry is the standard property for the kinetic theory description to be consistent with the second law of thermodynamics [5]. The reciprocal relations between different problems can be understood by the presented approach as a consequence of the Green reciprocity. Different from the entropy production argument, the thin layer that contains the surface of (or the interface with) the body inside is not included in the considered domain in our theory [11, 12]. This feature enables us to develop the theory for time-dependent problems [12] as a natural extension from the steady case [11]. As a consequence, a reinterpretation of the Kubo formulas (the fluctuation–dissipation theorem) for the viscosity and thermal conductivity is obtained. Further, mass and heat flow through a channel can be understood as a generalization of those formulas.

We shall show below the above mentioned results, following Refs. [11, 12]. However, we would like the reader to remember that some physical settings and notations here may be changed from the originals [11, 12] for conciseness.

Poiseuille flow and thermal transpiration will be repeated in the sequel as an illustrative example.

SYMMETRIC RELATION FOR STEADY BOUNDARY-VALUE PROBLEMS

Consider the steady behavior of a single-component monatomic rarefied gas in a domain \( \mathcal{D} \), where \( \mathcal{D} \) may be bounded or unbounded. The state of the gas is so close to the reference equilibrium state at rest with uniform density \( \rho_0 \) and temperature \( T_0 \) that the second (or higher) order effect of the deviation from this state can be neglected (linearization).

With a proper choice of reference length \( L \), we denote the spatial coordinates, molecular velocity, and velocity distribution function of molecules by \( \mathbf{x}, \mathbf{v}, \phi \). The linearized Boltzmann equation is given by

\[
\partial_t \phi = \sqrt{\pi Kn} \mathbf{L} \phi + I, \quad (\mathbf{x}, \mathbf{v} \in \mathbb{R}^3).
\]

(1)
Here Kn is the Knudsen number \(0 < Kn < \infty;\) \(Kn = \ell_0/L\) with \(\ell_0\) being the mean free path at the reference state) and \(I\) is a given function. Most typically, \(I\) represents the effect of a weak external force, though it is not limited to some real force (thus \(I\) will be referred to “external force” in the sequel). \(\mathcal{L}\) has the following well-known properties [15, 16]:

(i) \(\mathcal{L}(\Phi^+) = \mathcal{L}(\Phi^-)\), where \(\Phi^{-}(x, \xi) \equiv \Phi(x, -\xi)\). (commutative with a parity operation in \(\xi\)-space)

(ii) \(\langle \Phi, \mathcal{L}(\Psi) \rangle = \langle \Psi, \mathcal{L}(\Phi) \rangle\), where \(\langle \ldots \rangle = \int \ldots \langle \xi \rangle E(\xi) d\xi\). (self-adjointness)

(iii) \(\mathcal{L}(\Phi) = 0\) holds if and only if \(\Phi\) is a linear combination of \(1, \xi,\) and \(|\xi|^2\).

(iv) \(\langle \Phi, \mathcal{L}(\Phi) \rangle \leq 0\) on \(\mathcal{S}\) and the equality holds if and only if \(\Phi\) is a linear combination of \(1, \xi,\) and \(|\xi|^2\). (non-negativity)

Boundary \(\partial D\) of the domain \(D\) is classified into two parts: the interface with or surface of non-gas material (“body”), say real boundary \(\partial D_w\), and artificial boundary \(\partial D_a\), i.e., \(\partial D = \partial D_w \cup \partial D_a\). \(\phi\) obeys the following condition on \(\partial D_w\):

\[
\phi = g_w + \int_{\xi^* n_i < 0} \frac{\xi^* n_i |E(\xi^*)|}{\xi^* n_i |E(\xi)|} R(\xi^*, \xi; x) (\phi^* - g_w^*) d\xi^*, \quad \xi^* n_i > 0,
\]

where \(\phi\) and \(g_w\) with \(*\) indicate that their argument \(\xi\) is replaced by \(\xi^*\); \(n\) (or \(n_i\)) is a unit inward normal to the boundary; and \((2RT)^{1/2} u_w, T_0(1 + \tau_w),\) and \(p_0(1 + P_k)\) denote the velocity of the body surface, its temperature, and corresponding saturation pressure of the gas \((p_0 = \rho_k T_0/m)\). Because of the steady problem, \(D\) does not deform, so that \(u_w n_i = 0\). \(R\) represents the scattering law for gas molecules on the boundary at the reference state. \(R\) should satisfy standard properties in kinetic theory (such as the uniqueness condition [16], detailed balance [15, 5]), though there are some differences depending on whether \(\partial D_w\) is a simple boundary or condensed phase. The reader is referred to Appendix of Ref. [12] for the details. Here, we just remark that, when \(\partial D_w\) is a simple boundary, \(\langle \xi n_i \phi \rangle = 0\) and consequently \(P_w\) may be replaced by an arbitrary constant (for example, one may put \(P_w = 0\)).

Next consider the imaginary boundary \(\partial D_a\). We consider the imaginary boundary composed of, in most general, the following three parts \((\partial D_a^1 = \partial D_a^{(1)} \cup \partial D_a^{(2)} \cup \partial D_a^{(3)}))\), where \(\partial D_a^{(3)}\) represents the far distance when \(D\) is unbounded:

(i) On \(\partial D_a^{(1)}\), \(\phi\) obeys

\[
\phi(x, \xi) = h_{in}(x, \xi) \quad \text{for} \quad \xi n_i > 0, x \in \partial D_a^{(1)},
\]

where \(h_{in}\) is a given function for \(\xi n_i > 0\). For later discussions, we denote this function extended to the whole range of \(\xi\) by \(h(x, \xi)\). The way of extension is arbitrary and does not influence the results that follow.

(ii) On \(\partial D_a^{(2)}\), \(\phi\) obeys

\[
\phi = h(x, \xi) + \int_{\partial D_a^{(2)}} \int_{\xi' n'_i < 0} P(x', \xi', x, \xi) (\phi' - h') d\xi' dS' \quad \text{for} \quad \xi n_i > 0, x \in \partial D_a^{(2)},
\]

where \(h(x, \xi)\) is a given function for \(\xi \in \mathbb{R}^3, x \in \partial D_a^{(2)}; n'\) is the inward unit vector normal to \(\partial D_a^{(2)}\) at position \(x\); \(dS'\) is the surface element of \(\partial D_a^{(2)}\) at position \(x\); and \(\phi' = h(x, \xi')\). (iii) When \(D\) is unbounded, \(\phi\) obeys the following condition at a far distance (i.e., \(\partial D_a^{(3)}\)):

\[
\phi \rightarrow h(x, \xi) \quad \text{as} \quad |x| \rightarrow \infty,
\]

where \(h(x, \xi)\) is a solution of the Boltzmann equation (1).

The reader is referred to Appendix of Ref. [12] for the properties of \(P\) on \(\partial D_a^{(2)}\). Here we just remark that the typical example of \(P\) on \(\partial D_a^{(2)}\) is that \(\phi - h\) obeys the specular reflection or periodic condition. The most important is that the following relations hold on the boundary:

\[
\langle \xi n_i (\phi - g_w^\text{eq})(\phi^- - g_w^-) \rangle = 0 \quad \text{on} \quad \partial D_w, \quad \int_{\partial D_a^{(1)} \cup \partial D_a^{(2)}} \langle \xi n_i (\phi - h(\phi^- - h^-) \rangle dS = 0.
\]

\footnote{For example, \(\partial D_w\) is some control surface taken in a gas. Specular reflection and periodic boundary will also be classified into \(\partial D_a\).}
Now we are ready to state the symmetric relation for steady problems:

**Proposition 1** Consider the solutions \( \phi^A \) and \( \phi^B \) of the boundary-value problem (1)–(3) such that \( \phi^A \) (or \( \phi^B \)) is a solution in the case of \( I = I^A \), \( g_w = g^A_w \), and \( h = h^A \) (or \( I = I^B \), \( g_w = g^B_w \), and \( h = h^B \)), where \( \text{Kn} (0 < \text{Kn} < \infty) \), \( R \) in (2a), and \( P \) in (3b) are common to the problems. Then, if \( \partial D^S_3 \) is absent or \( \phi^A \) and \( \phi^B \) approach \( h^A \) and \( h^B \) in such a way that

\[
\int_{\partial D^S_3} \langle \zeta n_i (\phi^A - h^A) (\phi^B - h^B) \rangle dS = 0, \tag{5}
\]

the following equality holds:

\[
\int_{\partial D_w} \langle \zeta n_i g_B^A - \phi^A \rangle dS + \int_{\partial D_g} \langle \zeta n_i h^B - \phi^A \rangle dS - \frac{1}{2} \int_{\partial D_g} \langle \zeta n_i h^B - h^A \rangle dS - \int_D (\beta - \phi^A) d\mathbf{x}
\]

\[
= \int_{\partial D_w} \langle \zeta n_i g_B^A - \phi^B \rangle dS + \int_{\partial D_g} \langle \zeta n_i h^A - \phi^B \rangle dS - \frac{1}{2} \int_{\partial D_g} \langle \zeta n_i h^A - h^B \rangle dS - \int_D (\beta - \phi^B) d\mathbf{x}. \tag{6}
\]

It should be noted that the relation (6) holds unconditionally for bounded domain, while it is not clear whether or not the condition (5) for (6) to hold is fulfilled in general for unbounded domain. This is due to the slow decay of \( \phi - h \) at a far distance.\(^2\) Nevertheless,

1. for spatially one dimensional half-space problems,
2. for the case where \( \partial D_w \) is confined in a finite ball,

we can prove (5) and thus the relation (6) (see lemma 2 of Ref. [11]). For the other cases, one is required to check the condition (5) in each application of (6). Clarifying a general situation for (5) to hold would be desired.

In the relation (6), the role of problems A and B is symmetric. One of the important features of (6) is that each term is a definite flux of \( \phi^A \) or \( \phi^B \). It is due to (4) and (5) and implies that a proper flux through the boundary of concerned problem may be obtained from a flux of another problem. This observation eventually leads us to a general representation of flux through the boundary in terms of a properly defined Green function, which we shall explain in the next section. The relations between independent problems mentioned in the introduction are to be obtained as a consequence of that representation. [11, 13] Another feature of (6) is that \( I \) may be arbitrary and is not limited to the one arising from the linearization around local Maxwellian such as \( \zeta \) or \( \zeta (|\zeta|^2 - \frac{3}{2}) \). Actually, some unsteady problems and steady half-space problems can be formulated as the steady problem with other form of \( I \). Thus, for instance, relations between second-order slip coefficients can be discussed by the relation (6) (see Sec. 3 of Ref. [11]).\(^3\)

**GREEN FUNCTION AND GENERAL REPRESENTATION FOR FLUXES**

In this section, we introduce the Green function and discuss the general representation for the flux. Here we restrict ourselves to the simplest case that \( D \) is bounded, \( \partial D = \partial D_w \), and \( I = 0 \), for the clarity of main idea.\(^4\)

Taking into account the specific form of \( g_w \), let us consider the response of the system to the point source \( \alpha \) placed at \( \mathbf{x}_0 \) on \( \partial D_w \) and denote it by \( G^{(\alpha, \mathbf{x}_0)} (\mathbf{x}, \zeta) \), where \( \alpha = 1, 2 \zeta t_{1}(\mathbf{x}_0), |\zeta|^2 - \frac{3}{2} \) and \( t_{1}(\mathbf{x}_0) \) is a unit tangential vector to \( \partial D_w \) at \( \mathbf{x}_0 \). Thus, \( G^{(\alpha, \mathbf{x}_0)} (\mathbf{x}, \zeta) \) is a solution of the boundary-value problem (1) and (2) with \( I = 0 \) and \( g_w = \alpha \delta (\mathbf{x} - \mathbf{x}_0) \). Applying the symmetric relation (6) to the pair of \( G^{(\alpha, \mathbf{x}_0)} (\mathbf{x}, \zeta) \) and \( G^{(\beta, \mathbf{x}_1)} (\mathbf{x}, \zeta) (\beta = 1, 2 \zeta t_{1}(\mathbf{x}_1), |\zeta|^2 - \frac{3}{2}) \) leads to

\[
\langle \zeta n_i \beta^{-1} G^{(\alpha, \mathbf{x}_0)} (\mathbf{x}_1) \rangle = \langle \zeta n_i \alpha^{-1} G^{(\beta, \mathbf{x}_1)} (\mathbf{x}_0) \rangle. \tag{7}
\]

\(^2\) Note that, in three dimensional problem, according to Fourier’s law of heat conduction and Stokes equation for incompressible fluid, temperature and flow velocity approach their far field with the rate of \( 1/|\mathbf{x}| \) if \( \partial D_w \) is confined in a finite ball. Such behavior is properly discussed in Ref. [6], while much faster approach is claimed in Ref. [7] by assuming the local Maxwellian at a far distance. As pointed out in Refs. [11, 14], the fast approach estimate in Ref. [7] is not correct. Incidentally, there is no estimate on the far field behavior in Ref. [10].

\(^3\) In some problems, both side of (6) diverge due to the form of \( I \). In many cases, however, such divergence can be avoided by a proper transformation. In the reformulated problem, \( g_w \) may be changed to a more general form. In the case, in applying Proposition 1, \(- \frac{1}{2} \int_{\partial D_w} \langle \zeta n_i g_B^A - g_B^A \rangle dS \) should be added to the left-hand side and \(- \frac{1}{2} \int_{\partial D_g} \langle \zeta n_i g_B^A - g_B^A \rangle dS \) to the right-hand side of (6).

\(^4\) Ref. [11] gives a unified argument for two cases under \( I = 0 \): (i) \( \partial D_w = \partial D^S_1 \cup \partial D^S_2 \) (thus \( D \) is bounded) and (ii) \( \partial D_w = \partial D^S_3 \) (thus \( D \) is unbounded).
This means that the cause and result (induced flow or flux) are reciprocal between two different points on the boundary (Green reciprocity). Note that the solution of the same problem with arbitrary $g_w$ (and $I = 0$) can be expressed by

$$\phi(x, \zeta) = \int_{\partial D_w} \left( P_w(x_0) G^{(1)}(x, x_0) + |u_w(x_0)| G^{(2)}(\zeta u_w; x_0) (x, \zeta) + \tau_w(x_0) G^{(\beta)}(\zeta^2 - \frac{5}{2}; x_0) \right) dS_0,$$

where $dS_0$ denotes the element surface at position $x_0$ and $u_w$ is the unit vector in the direction of $u_w$. Thus, by taking into account (7), the mass, momentum, and heat flows across the boundary $\partial D_w$ at $x_1$ is expressed as

$$\langle \zeta n_i \beta \phi \rangle(x_1) = \int_{\partial D_w} \left( P_w(\zeta n_i G^{(\beta)}(\zeta^2 - \frac{5}{2}; x_1)) + w_s \langle 2 \zeta G^{(\beta)}(\zeta^2); x_1 \rangle \right) dS,$$

where $\beta = 1, 2 \zeta n_i (x_1)$, and $|\zeta|^2 - \frac{5}{2}$ gives mass-, momentum-, and heat-flow expression. This is the general representation of the fluxes through the boundary in the simplest case. Equation (8) shows that the flow (or flux) of $\beta$-moment at $x_1$ on $\partial D_w$ is immediately obtained if one knows the system response to the point source of $\beta$ at $x_1$. If (8) is integrated with respect to $x_1$ over a part of boundary $S(\subset \partial D_w)$, the mass and heat fluxes through $S$ is obtained as

$$\int_S \langle \zeta n_i \beta \phi \rangle dS = \int_{\partial D_w} \left( P_w(\zeta n_i G^{(\beta)}(\zeta^2 - \frac{5}{2}; x)) + w_s \langle 2 \zeta G^{(\beta)}(\zeta^2); x \rangle \right) dS,$$

where $\beta = 1, 2 \zeta n_i |\zeta|^2 - \frac{5}{2}$ and $G^{(\alpha,S)}(x, \zeta)$ is a solution of the boundary-value problem (1) and (2) with $I = 0$ and $g_w = \alpha \chi_S(x)$, where $\chi_S(x) = 1$ if $x \in S$ and $\chi_S(x) = 0$ otherwise.

Discussions about the above idea for wider class of situations are made in Ref. [11] (see footnote 4). For example, if $D$ is bounded and its boundary has an artificial part $\partial D_g$, the fluxes through arbitrary part $S$ of $\partial D_g$ are expressed as

$$\int_S \langle \zeta n_i \alpha \phi \rangle dS = \int_{\partial D_w} \langle \zeta n_i g_w \rangle dS + \int_{\partial D_g} \langle \zeta n_i \alpha \phi \rangle dS + \int_S \langle \zeta n_i \alpha \phi \rangle dS,$$

where $\alpha = 1, 2 \zeta n_i |\zeta|^2 - \frac{5}{2}$ and $G^{(\alpha,S)}(x, \zeta)$ is a solution of the boundary-value problem (1)–(3b) with $I = g_w = 0$ and $h = \alpha \chi_S(x)$. The reader is referred to Secs. 4 and 5 of Ref. [11] for more general representations and examples.

Poiseuille flow and thermal transpiration. Consider the steady Poiseuille flow and thermal transpiration in a straight pipe (Fig. 1). The geometry of the pipe cross-section may be arbitrary. Let us denote the perturbed velocity distribution function of the former by $\phi^P(x, \zeta)$ and the latter by $\phi^T(x, \zeta)$. Then $\phi^P$ is a solution of (1), (2), and (3b) in the domain grayed in Fig. 1 [0 ≤ $x_1$ ≤ 1 and $x_\perp \in S$, where $x_1 = (x_2, x_3)$] with $g_w = I = 0$ and $h = x_1$, while $\phi^T$ is a solution of (1), (2), and (3b) in the same domain with $g_w = h = x_1(|\zeta|^2 - \frac{5}{2})$ and $I = 0$ (magnitude of imposed gradient of pressure and temperature are rescaled as unity because of the linear problem; $P$ is the kernel of periodic condition). Note that $\phi^P$ is the Green function $G^{(1,S)}$ of the considered domain. Thus, from (10) with $\alpha = 1$, the following expression is eventually obtained (by taking account of obvious properties of $\phi^P$ and $\phi^T$):

$$\int_S \langle \zeta_1 \phi^T \rangle dS = \int_S \langle \zeta_1 (|\zeta|^2 - \frac{5}{2}) \phi^P \rangle dS.$$

This is the known reciprocity relation [3] as the Onsager–Casimir relation (see also Example 5 of Ref. [11]).

We finally mention that the relation of the above theory to entropy production argument is discussed in Ref. [14]. In Ref. [14], it is shown that the conventional Onsager–Casimir relation can be established in a pointwise way, thanks to the Green reciprocity. Further, the situation where the conventional Onsager–Casimir relation holds is clarified.\(^5\)

\(^5\) It is pointed out from early days that the conventional Onsager–Casimir relation does not hold in general, even for gross quantities like the fluxes discussed above. See, e.g., Sec. 15.10 of Ref. [5] and references therein. Thermophoresis is an example [6, 13, 14].
SYMMETRIC RELATION AND GREEN FUNCTION FOR UNSTEADY PROBLEMS

In discussing the reciprocity based on entropy production, it is necessary to consider the production in a thin layer that contains the interface between the gas and “body.” The production in the layer is estimated indirectly from the entropy balance at steady state by assuming the local equilibrium inside the body, which has naturally limited the argument to steady problems. On the other hand, the symmetric relation and Green function approach in previous sections do not contain the thin layer in the considered domain. In other words, the argument of the thin layer is not necessary to discuss the relations between two independent problems, once one accepts boundary conditions that have the standard properties in kinetic theory.

This is an extension of Proposition 1 to unsteady problems for fixed bounded domains. Symmetry of the role of problems A and B is retained thanks to the symmetric property of convolution. From the relation (14), one can derive a natural extension of the flux representations to unsteady problems. The reader is referred to Sec. 5 of Ref. [12] for the details. Here, we shall present some consequences obtained from this extension on the basis of Ref. [12].

With a proper reference scale \( t_0 \), we denote the time by \( t_0 t \). Below, \( \phi, I, g_w \) (or \( P_w, u_w \), and \( \tau_w \)), \( h_{\text{inv}} \), and \( h \) may depend on time: \( \phi(t, x, \zeta) \), \( I(t, x, \zeta) \), \( g_w(t, x, \zeta) \), \( P_w(t, x, \tau) \), \( u_w(t, x, \tau) \), and \( \tau_w(t, x, \tau) \), \( h_{\text{inv}}(t, x, \zeta, \tau) \), and \( h(t, x, \zeta, \tau) \). However, the domain \( D \) is bounded and does not change in time, so that \( u_w n_i = 0 \). Then, the behavior of the gas is described by the following linearized Boltzmann equation in place of (1):

\[
\text{Sh} \frac{\partial \phi}{\partial t} + \zeta \frac{\partial \phi}{\partial x_i} = \frac{2}{\sqrt{\pi} \text{Kn}} \mathcal{L}(\phi) + I \quad (\zeta \in \mathbb{R}^3, x \in D, t > 0). \tag{12}
\]

Here \( \text{Sh} = L/\text{Kn} \) is the Strouhal number. The initial data is denoted by putting the subscripted “initial”:

\[
\phi(0, x, \zeta) = \phi_{\text{initial}}(x, \zeta) \quad (\zeta \in \mathbb{R}^3, x \in D). \tag{13}
\]

Since \( D \) is bounded, \( \partial D_g(3) \) does not exist. The kernels \( R \) on \( \partial D_w \) and \( P \) on \( \partial D_s^{(2)} \) do not depend on \( t \). Then, for solutions of initial- and boundary-value problems (12), (13), (2)–(3b), we obtain the following:

**Proposition 2** Let \( \phi^A \) be a solution of the initial- and boundary-value problem (12), (13), (2)–(3b) with \( I = I^A, g_w = g_w^A, h = h^A, \) and \( \phi_{\text{initial}} = \phi_{\text{initial}}^A \). Let \( \phi^B \) be a solution of the initial- and boundary-value problem (12), (13), (2)–(3b) with \( I = I^B, g_w = g_w^B, h = h^B, \) and \( \phi_{\text{initial}} = \phi_{\text{initial}}^B \). Here the bounded domain \( D \), the Strouhal and Knudsen numbers \( \text{Sh} \) and \( \text{Kn} \), the collision operator \( \mathcal{L} \), and the kernels \( R \) and \( P \) are common to the problems of \( \phi^A \) and \( \phi^B \).

Then, the following relation holds:

\[
\text{Sh} \int_D \langle \phi_{\text{initial}}^A \phi^A(t, x) \rangle \text{d}x + \int_D \langle (t^{B*} - \phi^A) \rangle \text{d}x - \int_{\partial D_w} \langle \zeta n_i (g_{w^B} - \phi^A) \rangle \text{d}S
\]

\[
- \int_{\partial D_s} \langle \zeta n_i (h^{B*} - \phi^A) \rangle \text{d}S + \frac{1}{2} \int_{\partial D_s} \langle \zeta n_i (h^{B*} - h^A) \rangle \text{d}S
\]

\[
= \text{Sh} \int_D \langle \phi_{\text{initial}}^B \phi^B(t, x) \rangle \text{d}x + \int_D \langle (t^{A*} \phi^B) \rangle \text{d}x - \int_{\partial D_w} \langle \zeta n_i (g_{w^B} \phi^B) \rangle \text{d}S
\]

\[
- \int_{\partial D_s} \langle \zeta n_i (h^{A*} \phi^B) \rangle \text{d}S + \frac{1}{2} \int_{\partial D_s} \langle \zeta n_i (h^{A*} + h^B) \rangle \text{d}S. \tag{14}
\]

Here, \( f \ast g \) is a convolution of \( f \) and \( g \) with respect to time (thus \( f \ast g = g \ast f \)):

\[
f \ast g(t, \cdot) = \int_0^t f(r, \cdot) g(t - r, \cdot) \text{d}r.
\]

This is an extension of Proposition 1 to unsteady problems for fixed bounded domains. Symmetry of the role of problems A and B is retained thanks to the symmetric property of convolution. From the relation (14), one can derive a natural extension of the flux representations to unsteady problems. The reader is referred to Sec. 5 of Ref. [12] for the details. Here, we shall present some consequences obtained from this extension on the basis of Ref. [12].

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6 The appropriateness of the kinetic boundary conditions in view of the thermodynamics was studied by L. Waldmann, I. Kuščer, etc. See, for example, Refs. [1, 5, 8] and references therein.
Let us denote by $G^{(\alpha)}$ the response of the system to spatially uniform initial data $\phi_{\text{initial}} = \alpha(\zeta)$, where $\alpha$ is an arbitrary give function. Thus, $G^{(\alpha)}$ is a solution of initial- and boundary-value problem (12), (13), (2)–(3b) with $g_w = h = I = 0$ and $\phi_{\text{initial}} = \alpha$ (i.e., $G^{(\alpha)}_{\text{initial}} = \alpha$). Then, from (14), we obtain the following:

**Proposition 3** Consider the initial- and boundary-value problem (12), (13), (2)–(3b) in the domain $D$. Then the total $\alpha$-moment in the domain at any time are expressed in terms of the Green function for the initial data $\alpha$:

$$
\int_D \langle \alpha^E \phi \rangle(t, x) \, dx = \int_D \langle \phi_{\text{initial}}^E G^{(\alpha)} \rangle(t, x) \, dx + \int_D \langle I^E * G^{(\alpha)} \rangle(t, x) \, dx - \int_{\partial D} \langle \zeta n_i (g_w^E * G^{(\alpha)}) \rangle(t, x) \, dS - \int_{\partial D} \langle \zeta n_i (h^E * G^{(\alpha)}) \rangle(t, x) \, dS.
$$

(15)

When $\alpha = 1$, $\zeta$, and $\frac{2}{3}\|\zeta\|^2$, (15) gives the expression of the total mass, momentum, and energy in the domain.

**Poiseuille flow and thermal transpiration.** Consider the time-dependent Poiseuille flow and thermal transpiration (Fig. 1). The former can be formulated as the flow of a gas, which is initially in the thermal equilibrium at rest with the pipe wall of uniform temperature, caused by a uniform weak external force in the (negative) $x_1$-direction. Thus, $\phi^P$ is a solution of (12), (13), and (2) with $D = S$, $\partial D = \partial D_w = \partial S$, $I = -\zeta_1$, $g_w = 0$, and $\phi_{\text{initial}}^P = 0$ (the present $\phi^P$ with $t \to \infty$ is different from the previous steady one by $x_1$, though it does not influence the reciprocal relation here). On the other hand, $\phi^T$ of the thermal transpiration may be expressed by $\phi^T = x_1(\|\zeta\|^2 - \frac{5}{2}) + \Phi^T(t, x_\perp, \zeta)$, where $\Phi^T$ is a solution of (12), (13), and (2) with $D = S$, $\partial D_w = \partial S$, $\partial D_\| = 0$, $I = -\zeta_1(\|\zeta\|^2 - \frac{5}{2})$, $g_w = 0$, and a certain initial data $\Phi^T_{\text{initial}}(x_\perp, \zeta)$. Then, from (15) with $\alpha = \zeta$, the following expressions are obtained:

$$
\int_S \langle \xi_1 \phi^P \rangle(t, x_\perp) \, dx_\perp = -\int_S \langle \xi_1 G^{(\zeta_1)} \rangle(s, x_\perp) \, dx_\perp \, ds,
$$

(16a)

$$
\int_S \langle \xi_1 \phi^T \rangle(t, x_\perp) \, dx_\perp = -\int_S \langle \Phi^T_{\text{initial}} G^{(\zeta_1)} \rangle(t, x_\perp) \, dx_\perp - \int_S \langle \xi_1(\|\zeta\|^2 - \frac{5}{2}) G^{(\zeta_1)} \rangle(s, x_\perp) \, dx_\perp \, ds,
$$

(16b)

where the relation $u_1[\phi^T] = u_1[\Phi^T]$ has been taken into account. Thus, the mass flow (flux) through the pipe can be expressed for any time in terms of the Green function for the initial data $\zeta_1$. Further, since $G^{(\zeta_1)} \to 0$ as $t \to \infty$, (16) is reduced in the same limit to

$$
\int_S \langle \xi_1 \phi^T \rangle(t, x_\perp) \, dx_\perp = -\int_S \langle \xi_1(\|\zeta\|^2 - \frac{5}{2}) G^{(\zeta_1)} \rangle(s, x_\perp) \, dx_\perp \, ds.
$$

The left-hand side is the mass flow of the steady Poiseuille flow and thermal transpiration. These expressions will be interpreted as an extension of the fluctuation–dissipation theorem to the system of arbitrary $Kn$ in the next section.

Next we consider the response of the system to “external force” $I = \alpha(\zeta)$, which is uniform both in time and space, and denote it by $G^{(\alpha;I)}$, where $\alpha$ is an arbitrary function. Thus, $G^{(\alpha;I)}$ is a solution of initial- and boundary-value problem (12), (13), (2)–(3b) with $g_w = h = \phi_{\text{initial}} = 0$ and $I = \alpha$. Then, from (14), we obtain the following:

**Proposition 4** Consider the initial- and boundary-value problem (12), (13), (2)–(3b) in the domain $D$. Then, the total $\alpha$-moment in the domain from the initial time can be expressed in terms of the Green function for “external force” $\alpha$:

$$
\int_0^T \int_D \langle \alpha^E \phi \rangle(s, x) \, dx \, ds = \int_D \langle \phi_{\text{initial}}^E G^{(\alpha;I)} \rangle(t, x) \, dx + \int_D \langle I^E * G^{(\alpha;I)} \rangle(t, x) \, dx - \int_{\partial D} \langle \zeta n_i (g_w^E * G^{(\alpha;I)}) \rangle(t, x) \, dS - \int_{\partial D} \langle \zeta n_i (h^E * G^{(\alpha;I)}) \rangle(t, x) \, dS.
$$

(17)

When $\alpha = 1$, $\zeta$, and $\frac{2}{3}\|\zeta\|^2$, (17) gives the expression of total mass, momentum, and energy in the domain from the initial time.
Poiseuille flow and thermal transpiration. Consider again the time-dependent Poiseuille flow and thermal transpiration. As is seen from its definition, \( \phi^P \) is no other than \( G(-\zeta_1)(= -G^*(\zeta_1)) \). Thus, (17) with \( \alpha = -\zeta_1 \) and \( \phi = \Phi^t \) eventually leads to

\[
\int_D \langle \xi_1 \phi^T \rangle(s, x_\bot) \, dx_\bot \, ds = \text{Sh} \int_D \langle \Phi_{\text{initial}}^{-\alpha} \phi^P \rangle(t, x_\bot) \, dx_\bot + \int_D \langle \zeta_1 (|\zeta|^2 - \frac{5}{2}) \phi^P \rangle(s, x_\bot) \, dx_\bot \, ds,
\]

Equation (18b) is an extension of the reciprocal relation between the steady Poiseuille flow and thermal transpiration. In the limit \( t \to \infty \), the first term on the right-hand side vanishes, because \( \phi^P \) tends to a steady solution. Thus, in the same limit, (18b) is reduced to the known reciprocity that the (dimensionless) heat flux of the Poiseuille flow is identical to the mass flux of the thermal transpiration. It should be noted that the same reciprocity as the steady case remains valid for any time, if \( \Phi_{\text{initial}}^{-\alpha} = 0 \).

**SIMILARITY TO FLUCTUATION–DISSIPATION THEOREM**

The Green functions \( G^{(\alpha)} \) and \( G^{(\alpha,J)} \) in the previous section are closely related to each other. In fact, it is confirmed by substitution that \( \text{Sh}^{-1} \int_D G^{(\alpha)}(s,x,\zeta) \, ds \) solves the initial- and boundary-value problem for \( G^{(\alpha,J)} \):

\[
G^{(\alpha,J)}(t,x,\zeta) = \text{Sh}^{-1} \int_0^t G^{(\alpha)}(s,x,\zeta) \, ds.
\]

Thus, for two arbitrary functions \( \alpha(\zeta) \) and \( \beta(\zeta) \), the following identity holds:

\[
\int_D \langle \beta G^{(\alpha,J)} \rangle(t,x) \, dx = \text{Sh}^{-1} \int_0^t \int_D \langle \beta \rangle(t,x) \, dx \, ds.
\]

Equation (19) may be regarded as the counterpart of the fluctuation–dissipation theorem for bulk system or as its extension to the system of arbitrary Knudsen number (non-bulk system), which is understood as follows:

(i) Put \( \alpha = \beta \) and take the limit \( t \to \infty \). Then, the left-hand side of (19) represents the system response to “external force” \( \alpha \), while the right-hand side represents the self-correlation of the relaxation process from initial data \( \alpha \).

(ii) Applying (14) to the pair of \( G^{(\alpha)} \) and \( G^{(\beta)} \) gives

\[
\int_D \langle G^{(\alpha,J)} \rangle(t,x) \, dx = \int_D \langle G^{(\alpha)} \rangle(t,x) \, dx.
\]

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\[
\int_D \langle G^{(\alpha,J)} \rangle(t,x) \, dx = \int_D \langle G^{(\alpha)} \rangle(t,x) \, dx.
\]

where \( \alpha_\ell \) (or \( \beta_\ell \)) is 1 if \( \alpha \) (or \( \beta \)) is even and \(-1 \) if \( \alpha \) (or \( \beta \)) is odd. In the limit \( t \to \infty \), the integrand of both sides of (20) represents the static admittance at \( x \). In this sense, (21) represents an extension of the symmetry of static admittance [17, 18] to the system of arbitrary Kn.

It should be noted that the present approach and its consequence are more direct and stronger than those of Sec. 6 in Ref. [12]. We shall show below two examples that supplement the above observations.

Response of the gas in a periodic box. Let \( D \) be a periodic box and consider the system response \( G^{(2\zeta_1,\zeta_2)} \) against the “external force” \( \zeta = 2\zeta_1 \zeta_2 \). Obviously \( G^{(2\zeta_1,\zeta_2)} \) is spatially uniform and \( G^{(2\zeta_1,\zeta_2)} \to \left( \sqrt{\frac{2}{\pi}} \right) \zeta_1 \zeta_2 B(\zeta_1) \) as \( t \to \infty \). Here, \( B \) is the solution of \( \Delta' \zeta_2 B = -2\zeta_1 \zeta_2 \). It is well-known that \( B \) gives the viscosity \( \mu \) [16]:

\[
\mu = \sqrt{p_0 (2kT_0/m)^{-1/2}} t_0 (\zeta_1^2 \zeta_2^2 B).
\]

Now, as in (i), we take the limit \( t \to \infty \) in (19) with \( \alpha = \beta = 2\zeta_1 \zeta_2 \) to have

\[
\mu = p_0 \int_0^\infty \left( G^{(2\zeta_1,\zeta_2)}(t,\zeta_1,\zeta_2) \right) dt.
\]
In the same way, by considering the system response \( G^{(2\zeta_1,\zeta_2)} \) against the “external force” \( \alpha = \zeta_1(1|\xi|^2 - \frac{1}{2}) \), we eventually obtain the expression of thermal conductivity \( \lambda \) from (19):

\[
\lambda = 2(k/m)p_0 T_0 \int_0^\infty (G^{(\zeta_1(1|\xi|^2 - \frac{1}{2}))}_{\text{initial}}(\xi))(t,\xi) \, dt.
\]

These correspond to the so-called Kubo formula [17] in the linear response theory.

**Poiseuille flow and thermal transpiration.** We discuss again the time-dependent Poiseuille flow and thermal transpiration and give an alternative interpretation to the formulas in the previous sections. As already mentioned, \( \phi^p = G^{(-\zeta_1,\zeta_2)}(\zeta_1,\zeta_2)(= -G^{(\zeta_1,\zeta_2)}) \). Thus the expression (16a) is no other than (19) with \( \alpha = \beta = \zeta_1 \). Further, because \( \Phi^T \) is written as \( \Phi^T = G^{(\Phi^T_{\text{initial}})} + G^{(-\zeta_1(|\xi|^2 - \frac{1}{2}))}(= G^{(\Phi^T_{\text{initial}})} - G^{(|\xi|^2 - \frac{1}{2})) \), the expression (18a) with \( \Phi^T_{\text{initial}} = 0 \) is no other than (21) with \( \alpha = \zeta_1 \) and \( \beta = \zeta_1(1|\xi|^2 - \frac{1}{2}) \). Thus, the expressions (16a) and (18a) may be regarded as the extensions of the fluctuation-dissipation theorem and of the symmetry of static admittance respectively. In particular, if we take the limit \( t \to \infty \) for the latter, the reciprocity between the steady Poiseuille flow and thermal transpiration is recovered. This is an alternative explanation of the known reciprocity by the present approach.

**CONCLUDING REMARKS**

In this paper, we discussed the relations between two independent problems from the view point of Green function. One of the features of the presented approach enabled us to establish the reciprocity in a pointwise way (Green reciprocity), by which a general representation of fluxes is deduced. Another feature naturally enabled us to develop the theory for steady problems to unsteady problems. As a result, we obtained a natural extension (or counterpart) of the consequence of linear response theory for bulk system to the system of arbitrary Kn.

It is straightforward to develop the present theory [11, 14, 12] to gas mixtures. However, it should be noted that, in order to obtain the result for unbounded domain corresponding to Refs. [11, 14], it is necessary to prove a lemma corresponding to lemma 2 of Ref. [11]. This can be done by using that in a far field the concentration of component gas obeys the Laplace equation, in addition to that the flow velocity and temperature behave in the same way as in the corresponding to lemma 2 of Ref. [11].

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**REFERENCES**

11. S. Takata, J. Stat. Phys. 136, 751–784 (2009). (\( \frac{1}{2}\gamma_2 \) on line 18, p. 762 and line 24, p. 763 in this reference is a misprint of \( \frac{1}{2}\gamma_2 \).